Pergamon

Int. J. Solids Structures Vol. 34, No. 13, pp. 1547-1556, 1997

0020-7683/97 $17.00 + .00

PII: S0020-7683(96)00102-3

CONCEPT OF THE LIMIT YIELD CONDITION IN SHAKEDOWN THEORY

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(*Received* 31 *May* 1995; *in revised form* 4 *June* 1996)

Abstract—The concept of the limit yield condition is introduced which makes possible the extension of both the static (Melan) shakedown conditions, and the necessary kinematic (Koiter) one to a class of classical constitutive material models with internal variables. This class includes material models with both bounded and unbounded nonlinear isotropic strain-hardening. It is assumed that the yield conditions are convex with respect to stresses for all admissible values of the internal variables, but convexity in the internal variables is not assumed. Connections are established between the response of elastic perfectly plastic bodies to cyclic loading and that of bodies with internal variables. A method for estimating the limit yield condition is developed, and an example of this application is given. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

It is well known that the basis of modern shakedown theory was originally developed by Melan (1938) and Koiter (1960) in the context of elastic perfectly plastic bodies. Many efforts have been devoted during the last decades to extending the shakedown theory to more realistic material behavior. These efforts have been concerned mainly with the strain-hardening phenomenon. Melan (1938) was the first to present the static shakedown theorems for the material model with linear unlimited kinematic strain hardening. Similar assumptions were taken by Maier (1973) and Ponter (1975). König (1982) and König and Siemaszko (1988) considered nonlinear kinematic hardening. Another approach was developed by Maier and Novati (1990a, b) who modeled the general hardening of materials by piece-wise linear yield surfaces. A limited linear kinematic hardening rule was considered by Weichert and Gross-Weege (1988). They used a specific two surface yield condition for this purpose. Stein *et al.* (1992) extended the static shakedown theorem to general nonlinear kinematic hardening phenomena employing the so called overlay constitutive material model.

Invention of the generalized standard material model (GSMM) (Halphen and Nguyen, 1975) provided new opportunities for extensions of the shakedown theory to more general material models. Mandel (1976) used GSMM to extend the static shakedown theorem to linear and unbounded strain-hardening. Polizzotto *et al.* (1991) employed GSMM to demonstrate how the effects of the material microstructure can be incorporated into the shakedown theory. In particular, they extended the kinematic (Koiter) shakedown theorems to GSMM.

Corigliano *et al.* (1995) developed kinematic approach to the problems of dynamic shakedown. To that end they employed the so called generalized nonstandard material model (GNMM) with nonassociativity and bounded nonlinear hardening and the assumption of convexity of plastic potential in stresses and internal variables.

Earlier the problems of dynamic shakedown was considered by Comi and Corigliano (1991) who employed GSMM to extend both static and kinematic shakedown theorems to strain-hardening structures.

The sufficient static (Melan) shakedown condition was extended to the GNMMs by Pycko and Maier (1995) under the same assumptions. Hachemi and Weichert (1992) extended the shakedown theory to account for effects of material damage and linear kinematic strain-hardening. They utilized an extension of GSMM to damaged materials developed by Ju (1989).

Recently Nayroles and Weichert (1993) put forward the notion of "elastic sanctuary" which may be applied to some situations considered below (see Sections 7, 8). It is worth noting that if the sanctuary exists no condition of convexity of the real yield surface is needed for the proof of the static shakedown theorem. A similar notion of "reduced elastic domain" was advanced by Maier (1969).

All known extensions of the shakedown theory to strain-hardening bodies were obtained under various assumptions, some of which seem to be unnecessary. In particular, this remark relates to GSMM which assumes that the yield function is the potential for the thermodynamical forces corresponding to the internal variables. This assumption imposes a strong limitation on the possibility to apply GSMM to real materials. Moreover, to extend the shakedown theory to GSMM it is necessary to assume that the yield function is convex with respect to internal variables.

Shakedown of elastic-perfectly plastic bodies (briefly: perfect bodies) is known to occur due to formation of certain fields of residual stresses, whereas the shakedown of bodies with more complex mechanical behavior occurs not only due to residual stresses but also due to changes in their microstructure, for example, due to strain-hardening. These changes, being reflected in the evolution of appropriate chosen internal parameters, result in changes in the yield condition.

The approach developed in this paper is based on the concept of the limit yield conditions (LYC): the yield condition corresponding to the post-adaptation stage of the deformation process. This stage has attracted the attention of investigators for a long time (Martin, 1975: Polizzotto, 1994; Gokhfeld & Sadakov, 1995). A preliminary consideration of some questions touched in this paper has been done by Druyanov and Roman (1995). The concept of LYC allows one to compare the response of perfect bodies to cyclic loading with that of non perfect ones and to establish connections between them. It is shown that this concept is valid for both elastic and asymptotic shakedown. The necessary static (Melan) condition for shakedown is extended to the classic constitutive material models with internal variables. From this, the necessary kinematic (Koiter) shakedown condition is also extended to non-perfect bodies.

The usual assumption of convexity of the yield function (which is taken as the plastic potential) with respect to the stress tensor components is made, but it is not assumed to be convex in the internal variables. Based on this assumption the sufficient static (Melan) shakedown conditions can be extended, under certain conditions, to non-perfect bodies, in particular, to bodies with both bounded and unbounded nonlinear isotropic strain-hardening.

There is a certain difference between the mechanical behavior of bodies with internal variables (briefly: non perfect bodies) and perfect bodies. The former do not suffer any plastic deformation during neutral loading (loading along the yield surface), whereas the perfect bodies undergo plastic deformation only under neutral loading. Therefore, if a perfect body shakes down elastically, the stress path will be located either completely within the interior of the yield surface from some time on, or can touch it at some instants. In contrast to this behavior, the stresses can satisfy the LYC during some time if elastic shakedown of a non-perfect body takes place.

Nevertheless, the comparison of mechanical behavior of perfect bodies with behavior of non-perfect ones makes possible the development of methods to estimate the response of non perfect bodies to cyclic loading. A method capable of evaluating the LYC and the corresponding values of the internal parameters is proposed and examples of its application are considered.

2. CONSTITUTIVE MATERIAL MODEL

Changes in the material microstructure due to plastic deformation can be taken into account in the framework of the classical constitutive material models by exploiting the concept of internal variables. It is assumed that the components of the strain tensor $\boldsymbol{\varepsilon}$ are small, i.e. they are linear with respect to the components of the displacement.

Let $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ be the yield condition, $\boldsymbol{\sigma}$ be the stress tensor and $\boldsymbol{\chi}$ be the set of internal variables which is formally considered here as a vector. The components of the plastic strain tensor $\boldsymbol{\varepsilon}^{\mathrm{p}}$ can be included in the set $\boldsymbol{\chi}$. The yield function $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi})$ is taken as the plastic potential.

The constitutive equations may be written as follows:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \tag{1}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\lambda}\Phi_{,\boldsymbol{\sigma}}, \quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \mathbf{L} : \dot{\boldsymbol{\sigma}} \tag{2}$$

$$\dot{\lambda} > 0 \quad \text{if } \Phi = 0 \quad \text{and} \quad \mathrm{d}'\Phi = \Phi_{,\boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{\sigma} > 0 \tag{3}$$

$$\dot{\lambda} = 0 \quad \text{if } \Phi < 0, \quad \text{or if } \Phi = 0 \quad \text{and} \quad \mathrm{d}'\Phi \leqslant 0 \tag{4}$$

$$\dot{\boldsymbol{\chi}} = \mathbf{B} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \tag{5}$$

where $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ are the elastic and plastic parts of the strain rate tensor $\dot{\boldsymbol{\varepsilon}}$, $\dot{\lambda}$ is a non-negative scalar factor, $\mathbf{L}$ is the tensor of elastic compliance. $\mathbf{B}(\boldsymbol{\sigma}, \boldsymbol{\chi})$ is a third-order tensor responsible for the evolution of the internal variables, $\Phi_{,\boldsymbol{\sigma}} = \partial\Phi/\partial\boldsymbol{\sigma}$, $\dot{\varepsilon} = \mathrm{d}\varepsilon/\mathrm{d}t$, $\mathbf{a} : \mathbf{b} = \mathrm{a}_{ij}\mathrm{b}_{ij}$, $\mathbf{B} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \mathrm{B}_{ikl}\varepsilon^{\mathrm{p}}_{kl}$ $\mathrm{B}_{ikl} = \mathrm{B}_{ilk}$. It is assumed that the material is stable, i.e. it cannot lose its strength during a loading process. See eqns (3), (4).

Consider the equation $\dot{\Phi} = \Phi_{\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \Phi_{,\boldsymbol{\chi}} \cdot \dot{\boldsymbol{\chi}} = 0$. An application of eqns (2) and (5) yields $\Phi_{,\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + \dot{\lambda}\Phi_{,\boldsymbol{\chi}} \cdot \mathbf{B} : \Phi_{,\boldsymbol{\sigma}} = 0$. This equation allows to determine $\dot{\lambda}$ and exclude it from (2)–(4) if $\Phi_{,\boldsymbol{\chi}} \cdot \mathbf{B} : \Phi_{,\boldsymbol{\sigma}} \neq 0$. During neutral loading $\Phi_{,\boldsymbol{\sigma}} : \boldsymbol{\sigma} = 0$ and $\dot{\lambda} = 0$ if $\Phi_{,\boldsymbol{\chi}} \cdot \mathbf{B} : \Phi_{,\boldsymbol{\sigma}} \neq 0$. Thus, the material does not suffer any plastic deformation in this case. Let $\boldsymbol{\eta}(\boldsymbol{\sigma}, \boldsymbol{\chi},)$ be the vector of thermodynamical forces dual to $\boldsymbol{\chi}$. The specific rate of dissipation is, $\dot{\mathrm{D}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\eta} \cdot \dot{\boldsymbol{\chi}}$ where $\boldsymbol{\eta} \cdot \boldsymbol{\chi} = \eta_i \dot{\chi}_i$. According to the second law of thermodynamics, $\dot{\mathrm{D}} \geqslant 0$ during actual deformation processes.

Specify the yield function such that $\Phi < 0$ corresponds to the interior of the yield surface $\Phi = 0$, i.e. to the elastic part of the stress space $\boldsymbol{\sigma}$. Suppose also that the point $\boldsymbol{\sigma} = 0$ is always in the interior of the yield surface, i.e. $\Phi(0, \boldsymbol{\chi}) < 0$. Assume that the yield condition is convex (not concave) with respect to $\boldsymbol{\sigma}$ for all admissible values of its arguments. Any convexity of the yield function in the internal variables is not supposed.

Consider the yield condition $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ for a fixed value of $\boldsymbol{\chi}$. The convexity of the yield condition in stresses results in the known inequality which holds, according to the assumption, for any $\boldsymbol{\sigma}$ satisfying the yield condition $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$:

$$(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}) : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geqslant 0, \tag{6}$$

where $\hat{\boldsymbol{\sigma}}$ is a statistically admissible stress tensor i.e. it satisfies the yield inequality

$$\Phi(\hat{\boldsymbol{\sigma}}, \boldsymbol{\chi}) \leqslant 0. \tag{7}$$

If the yield function is strictly convex with respect to $\boldsymbol{\sigma}$ the sign of equality in (6) holds only if $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$, or if $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = 0$. For $\hat{\boldsymbol{\sigma}} = 0$ the inequality (6) yields: $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \geqslant 0$, i.e. the rate of plastic work ($\dot{\mathrm{D}}^{\mathrm{p}}$) is non-negative.

3. LIMIT YIELD CONDITION

In this and following sections an elastic body subjected to cyclic quasi static loading is considered. The loads may be of static and/or kinematic nature and are given by prescribed surface reactions $\mathbf{P}(t)$ at the part S_{p} of the body surface, or by prescribed surface velocity vectors $\mathbf{V}(t)$ at the part S_{v}.

Two paths to plastic failure under cyclic loading are usually considered, ratchetting (unlimited accumulation of plastic deformation of the same sign) and plastic shakedown (non-decreasing alternating plastic deformation). In both cases the total plastic work is unbounded: $D^p \to \infty$ as $t \to \infty$, where

$$D^p = \int_0^t dt \int_Q \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p \, dQ, \tag{8}$$

and Q denotes the part of space occupied by the body. If D^p is bounded from above ($D^p < \infty$) the body is said to shake down (adapt itself) to the prescribed cyclic loading program. Note that $D^p \geqslant 0$ because $\dot{D}^p \geqslant 0$. Obviously, this identification excludes the plastic shakedown out of consideration. According to the accepted constitutive model, any plastic deformation causes changes in the internal variables (eqn (5)) which in turn affect the yield function. If irreversible changes in the material microstructure cease in a limited time (t^*), the shakedown is called elastic. In this case, only elastic deformation occurs from time t^* on and the entire deformation process can be divided into transient and stationary (with respect to the internal variables), or post-adaptive stages. The duration of the transient stage (t^*) is called the adaptation time.

If the elastic shakedown occurs, the internal variables approach definite limit values $\boldsymbol{\chi}^*$ as $t \to t^*$ and consequently one may speak of a limit yield condition (LYC): $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}^*) = 0$. It is necessary to assume that the adaptation time may be infinite ($t^* = \infty$), although no examples of such behavior are known. This kind of shakedown my be called asymptotic. In this case, the adaptation state is not reached in a finite time, however, as distinct from the case of inadaption, the plastic work is bounded: $D^p < \infty$.

The stresses do not approach zero as $t \to \infty$ because they must satisfy the yield condition. Hence, the plastic dissipation is bounded from above ($D^p < \infty$) if strain rate tensor components tend to zero ($\dot{\varepsilon}^p \to 0$) like $t^{-(1-\alpha)}$ as $t \to \infty$, where $\alpha > 0$. Therefore, the components of the plastic strain tensor have definite limits as $t^* \to \infty$. Owing to eqn (5), $\boldsymbol{\chi} \to 0$ like $\dot{\varepsilon}^p$ as $t^* \to \infty$. Therefore, under the natural assumption that the thermodynamical forces $\boldsymbol{\eta}$ are bounded, $\boldsymbol{\chi}$ has a definite limit in this case. Thus, a LYC exists in the case of asymptotic shakedown as well.

The notion of the asymptotic shakedown holds not only for non-perfect bodies but for perfect ones as well. The existence of the LYC can be taken as a new definition of shakedown for elastic non-perfectly plastic bodies.

4. NECESSARY STATIC CONDITION FOR ELASTIC SHAKEDOWN

In the case of elastic shakedown, the actual stresses do not cause any plastic deformation from some time on and a LYC exists. Remember that the actual stresses can be expanded as a sum: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^r$, where $\boldsymbol{\sigma}^E(t)$ denotes the stresses that would occur in the body under the current external loads if the material behavior of the body were purely elastic and $\boldsymbol{\sigma}^r$ denotes the residual stresses. $\boldsymbol{\sigma}^E(t)$ is determined by solution of the elastic body value problem for $\mathbf{P}(t)$ and $\mathbf{V}(t)$.

If a body shakes down elastically it experiences only elastic deformation from some time on. In linear elasticity, existing initial stresses are not affected by current stresses and strain. Therefore, after plastic deformation has ceased the residual stresses (which should be considered as the initial ones for the post-adaptive stage of the deformation process) and the internal variables do not change.

Thus, a necessary condition for shakedown of elastic non-perfectly plastic bodies may be formulated as follows: if an elastic non-perfectly plastic body shakes down elastically, then there exists a field of residual stresses ($\hat{\boldsymbol{\sigma}}^r$) and set of internal variables ($\hat{\boldsymbol{\chi}}$), both independent on time, such that the stress tensor $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^E(t) + \hat{\boldsymbol{\sigma}}^r$ satisfies the yield inequality

$$\Phi(\hat{\sigma}, \hat{\chi}) \leqslant 0 \tag{9}$$

for any values of loads from some time $\hat{t}$ on, where equality holds only if

$$\Phi_{,\hat{\sigma}} : \hat{\sigma} = 0. \tag{10}$$

This condition is the slightly specified Melan necessary condition for shakedown. It will be shown in Section 9 that in the event of isotropic strain hardening, the above condition is a sufficient one as well.

Remark. Repeating the same arguments for perfect plastic bodies one comes to the conclusion that if an elastic perfectly plastic body shakes down elastically then there exists a field of time-independent residual stresses ($\hat{\sigma}^r$) such that the stress tensor $\hat{\sigma} = \sigma^E(t) + \hat{\sigma}^r$ satisfies the yield inequality $\Phi(\hat{\sigma}, \hat{\chi}) \leqslant 0$, where equality holds at some isolated instants.

5. SUFFICIENT KINEMATIC CONDITION FOR ELASTIC INADAPTATION

This condition was developed by Koiter (1960) in the framework of the theory of perfectly plastic bodies. Let $\dot{W}_a(t)$ denote the rate of work of external loads and $\dot{D}_a^p(t)$ do the rate of specific plastic work corresponding to an admissible cycle of plastic strain rates (AC):

$$\dot{W}_a(t) = \int_{S_p} \mathbf{P} \cdot \mathbf{v}_a \, \mathrm{d}Q, \quad \dot{D}_a^p(t) = \int_Q \sigma_a : \dot{\varepsilon}_a^p \, \mathrm{d}Q, \tag{11}$$

where $\varepsilon_a^p(t)$ denotes an AC and S_p is the part of the body surface where external traction $\mathbf{P}(t)$ is prescribed. The notion of AC is taken in accordance with its classical definition; $\dot{\varepsilon}_a^p(t)$ has to satisfy the compatibility equations and the velocity vector field $\mathbf{v}_a$ corresponding to $\dot{\varepsilon}_a^p$ has to satisfy zero kinematic boundary conditions at the part of body surface where kinematic boundary conditions are prescribed; $\mathbf{v}_a = 0$ at S_v.

In eqn (11), $\sigma_a(t)$ denotes the stress tensor corresponding to $\dot{\varepsilon}_a^p(t)$ in accordance with the associated flow rule. Let W_a denote the external work for AC and D_a do the plastic work for the same period:

$$W_a = \int_{t_0}^{t_0 - T} \dot{W}_a \, \mathrm{d}t, \quad D_a = \int_{t_0}^{t_0 - T} \dot{D}_a \, \mathrm{d}t. \tag{12}$$

T denotes the duration of the cycle and t_0 does an arbitrary time. The kinematic (Koiter, 1960) inadaptation theorem (necessary shakedown condition for perfectly plastic bodies) asserts, shakedown is impossible if there exists a program of loading $\mathbf{P}(t)$ taken from a prescribed range of loads and an admissible cycle of plastic strain rates $\dot{\varepsilon}_a^p(t)$ such that the inequality is valid

$$W_a > D_a. \tag{13}$$

As applied to the non-perfect bodies the sufficient kinematic condition of inadaptation can be formulated as follows: a set of internal variables $\hat{\chi}$ cannot be the limit one for a given loading program $\mathbf{P}(t)$ if there exists an AC such that inequality (13) is valid for the yield condition corresponding to $\hat{\chi} = \text{const.}$ from some time $\hat{t}$ on, that is for $t_0 > \hat{t}$.

It is necessary to underline that the proposed condition does not assert that the body under consideration cannot shakedown to the given loading program in general, or that $\hat{\chi}$ cannot be the limit set of the internal variables for another loading program from the prescribed bounds. It only answers the question, could the set of internal variables $\hat{\chi}$ be the limit one for the given loading program.

Thus, the yield condition to verify is known in advance, $\Phi(\sigma, \hat{\chi}) = 0$. This circumstance makes it possible to employ the classical scheme (Koiter, 1960) to prove the above assertion.

Assume for a while that the assertion to prove is not correct, that is, the body can adapt itself to the loading program $\mathbf{P}(t)$ and $\hat{\chi}$ is the limit set of the internal variables. Then, according to the necessary shakedown condition for not perfect bodies, there exists a field of time-independent residual stresses $\hat{\sigma}^r$ such that stress field $\hat{\sigma} = \sigma^E + \hat{\sigma}^r$ satisfies inequality $\Phi(\hat{\sigma}, \hat{\chi}) \leqslant 0$ and does not cause deformation from some time $\hat{t}$ on. Repeating Koiter's arguments (Koiter, 1960) one can derive the equality:

$$W_a = \int_{t_0}^{t_0+T} \mathrm{d}t \int_Q \hat{\sigma} : \dot{\varepsilon}_a^p \, \mathrm{d}Q.$$

Due to assumed convexity of the yield condition $\hat{\sigma} : \dot{\varepsilon}_a^p \leqslant \sigma_a : \dot{\varepsilon}_a^p$. Therefore $W_a \leqslant D_a$, which is in conflict with eqn (13). Q.E.D.

Obviously, the strict convexity of the yield condition is not necessary in order for the above consideration to be valid. It is enough to assume that the yield condition is not concave.

6. SUFFICIENT STATIC CONDITION FOR SHAKEDOWN

It is assumed now that the yield condition is strictly convex with respect to the stress tensor components. Let $\chi(t)$ denote the current actual set of internal variables. A sufficient condition for shakedown may be formulated as follows: if there exists a time-independent field of residual stresses $\hat{\sigma}^r$ such that the stress tensor $\hat{\sigma}(t) = \sigma^E(t) + \hat{\sigma}^r$ is safe, that is, it satisfies the yield inequality

$$\Phi(\hat{\sigma}(t), \chi(t)) < 0 \qquad (14)$$

from some time on, then the body will shake down elastically to the given loading program.

Inequality (14) holds if $\hat{\sigma}$ is safe for any current actual yield conditions starting with some time $\hat{t}$. This implies that $\hat{\sigma}$ is in the interior of the current yield surface from this time on. Sometimes the validity of inequality (14) can be checked in advance, without a calculation of the entire actual deformation path. An example is the bodies with isotropic strain hardening (Section 9). In some cases the proposed theorem provides an opportunity to develop methods for approximate estimating of the LYC (Section 8).

The proof of the theorem follows the known scheme (See Koiter, 1960). However, to complete the question let us repeat the main points of the proof and make sure that under the given conditions, the theorem is valid not only for perfect bodies but for non-perfect ones as well. Let σ, $\dot{\varepsilon}$, χ, $\sigma^r, \ldots,$ be the corresponding actual current values of the field variables.

Consider the elastic energy corresponding to the difference in residual stresses $\sigma^r, \hat{\sigma}^r$.

$$W = \frac{1}{2}\int_Q (\sigma^r - \hat{\sigma}^r) : \mathrm{L} : (\sigma^r - \hat{\sigma}^r) \, \mathrm{d}Q. \qquad (15)$$

Because $\hat{\sigma}^r$ is time-independent from $\hat{t}$ on, the derivative of W with respect to time is

$$\dot{W} = \int_Q (\sigma^r - \hat{\sigma}^r) : \dot{\varepsilon}^{re} \, \mathrm{d}Q. \qquad (16)$$

where $\dot{\varepsilon}^{re}$ is the actual elastic strain rate tensor corresponding to $\dot{\sigma}^r$. This is the sole spot in the proof where the assumption: $\hat{\sigma}^r$ is time independent is utilized. The rest of the proof does not need any comments.

The above sufficient condition for shakedown coincides with conditions established by certain authors for non-perfect plastic bodies under various assumptions. For references see Stein *et al.* (1992), Polizzotto *et al.* (1991), Hachemi and Weichert (1992). However, unlike them, the above condition is not based on any additional assumptions except for the assumption that a time independent residual stress field which satisfies inequality (14) exists from some time on.

7. SOME THEOREMS RELATING TO SHAKEDOWN OF ELASTIC PERFECTLY PLASTIC BODIES

Consider two geometrically identical elastic perfectly plastic bodies B_1 and B_2 under the same kinematic constraints, subjected to the same loading program however with different yield conditions $F_1(\boldsymbol{\sigma}) = 0$ and $F_2(\boldsymbol{\sigma}) = 0$ correspondingly. As previously, let F_1 and F_2 be defined in such a way that inequalities $F_1 < 0$ and $F_2 < 0$ correspond to the interiors of the corresponding surfaces. Assume that for all admissible values of $\boldsymbol{\sigma}$ the inequality is valid

$$F_1(\boldsymbol{\sigma}) > F_2(\boldsymbol{\sigma}). \tag{17}$$

This assumption implies that the surface $F_2 = 0$ encloses the surface $F_1 = 0$. Let $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ be admissible stresses for both yield conditions $F_1 = 0$ and $F_2 = 0$—correspondingly, that is, they satisfy the inequalities $F_1(\boldsymbol{\sigma}_1) \leqslant 0$ and $F_2(\boldsymbol{\sigma}_2) \leqslant 0$. Two assertions can be delivered:

(1) If shakedown takes place for B_1 than it takes place for all the bodies with the yield surfaces enclosing $F_1 = 0$. According to the necessary static shakedown theorem, there exists a residual stress field $\hat{\boldsymbol{\sigma}}^r$ such that the stress $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}^r + \boldsymbol{\sigma}^E(t)$ is admissible for the body B_1, that is, the inequality is valid $F_1(\hat{\boldsymbol{\sigma}}) \leqslant 0$. According to eqn (17), $F_2(\hat{\boldsymbol{\sigma}}) < 0$. Now the sufficient static condition of shakedown for perfect bodies (the Melan theorem) yields that shakedown takes place or B_2 as well. It is necessary noting that, according to Nayroles and Weichert (1993) the surface $F_1(\boldsymbol{\sigma}) = 0$ may be considered as the elastic sanctuary.

(2) If shakedown is impossible for B_2 it is impossible for any body with the yield surface enclosed by surface $F_2 = 0$. Otherwise shakedown would be possible for B_2. The yield condition for which the body shakes down to the prescribed loading program, may be called the "adaptive yield conditions". On the contrary, a yield condition for which shakedown is impossible is called here the "inadaptive yield conditions".

The set of adaptive yield conditions is denoted A, the set of inadaptive yield conditions is $\bar{A}$.

8. ESTIMATING OF LIMIT YIELD CONDITION

Some connections between mechanical behavior of perfect and non-perfect bodies under cyclic loading can be established. To that end, let us consider a non-prefect body B with the yield condition $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$, as well as a perfect body B_p with the yield condition $F(\boldsymbol{\sigma}) = 0$ of the same shape, under the same kinematic constraints, and subjected to the same loading program as B. Two following assertions are valid:

(1) Let $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \leqslant F(\boldsymbol{\sigma})$ for $t \geqslant t^*$. Assume that the body B_p shakes down to a loading program. Then the body B will shake down to the program as well. Really, because the body B_p shakes down a time independent residual stress field $\hat{\boldsymbol{\sigma}}^r$ exists such that the stress $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}^r + \boldsymbol{\sigma}^E(t)$ is either completely in the interior of the surface $F(\boldsymbol{\sigma}) = 0$, or touches it at some instants, from a time t^* on. This stress yield is safe with respect to the surfaces $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ for $t > t^*$. According to the sufficient static condition for shakedown (Section 6) the body B will shake down to the loading program. Obviously, the surface $F(\boldsymbol{\sigma}) = 0$ is a sunctuare in respect to the surfaces $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ for $t \geqslant t^*$ (Neyroles and Weichert, 1993).

(2) Consider the opposite situation. Let the inequality $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) \geqslant F(\boldsymbol{\sigma})$ be valid, i.e. the surfaces $\Phi(\boldsymbol{\sigma}, \boldsymbol{\chi}) = 0$ are enclosed by the surface $F(\boldsymbol{\sigma}) = 0$, or coincide with it. Then, if the body B_p does not shake down to the given loading program, then B does not shake down as well, because under such circumstances the necessary static shakedown condition for the both is not satisfied.

The above theorems allow to estimate the LYC by means of the comparison of the mechanical behavior of non perfect bodies with that of perfect bodies.

9. ISOTROPIC STRAIN HARDENING

The bodies with isotropic stain hardening are an example of the bodies to which the above mentioned methods are applicable. In that event the yield condition depends on a sole non decreasing scalar parameter χ. The yield surfaces with greater values of χ embrace the surfaces with lesser values, and the yield functions with greater values of χ are less than the yield functions with lesser values: $\Phi(\boldsymbol{\sigma}, \chi_2) < \Phi(\boldsymbol{\sigma}, \chi_1)$ if $\chi_1 < \chi_2$.

First of all consider the event that the strain hardening is bounded $\chi < X = \text{const}$. Suppose that there exists a value of the hardening parameters $\hat{\chi} < X$ for which a time independent residual stress field $\hat{\boldsymbol{\sigma}}^{\mathrm{r}}$ can be found such that the stress field $\hat{\boldsymbol{\sigma}}(t) = \boldsymbol{\sigma}^{\mathrm{e}}(t) + \hat{\boldsymbol{\sigma}}^{\mathrm{r}}$ is safe starting with some time. This stress field is safe for all the yield conditions corresponding to $\chi > \hat{\chi}$.

Under such a condition the body will shake down to the prescribed loading program. Really, during the course of deformation the actual values of χ will exceed $\hat{\chi}: \chi > \hat{\chi}$ starting with some time (otherwise the body would shake down to a value of χ less χ^*). Now the sufficient condition of shakedown (Section 6) is applicable, and one can conclude that the body under consideration will shake down to the taken loading program.

If strain-hardening is unbounded, the body shakes down to any bounded loading program because it is always possible to find so large value of the strain-hardening parameter ($\hat{\chi}$) that the applied loads simply will not cause a plastic deformation. Thus, for the bodies with isotropic strain-hardening the classic necessary shakedown condition (Section 4) is at the same time a sufficient one.

At the post adaptive stage, stresses reach the level of limit yield surface, but do not cause plastic deformation. This is possible if the stress path is tangent to the limit yield surface, or only touches it at some instants. This conditions can be written as $\Phi_{,\sigma}\dot{\boldsymbol{\sigma}} = 0$. The hardening parameter grows until any plastic deformation ceases, that is, as long as the stresses satisfy the current yield surface and $\Phi_{,\sigma} : \dot{\boldsymbol{\sigma}} > 0$.

Consider the yield conditions corresponding to different values of χ as the yield conditions of perfect bodies with the different yield conditions. Suppose that the subsets of adaptive (A) and inadaptive ($\bar{\mathrm{A}}$) surfaces are not empty. The interfacial yield is minimum in A and maximum in $\bar{\mathrm{A}}$. Let χ_0 be the value of hardening parameter corresponding to this surface and χ^* corresponding to the LYC.

No surface from $\bar{\mathrm{A}}$ can be the limit one. Thus, $\chi^* \geqslant \chi_0$. This inequality provides a lower estimation of the limit value of the hardening parameter, χ^*. Any yield surface from A can be the limit one. Which of them is the actual limit yield surface depends on accumulated plastic deformation, that is, on deformation path. The interfacial yielding surface corresponds to the minimal value hardening parameter.

10. EXAMPLE

Consider a structure loaded by a variable P ranging from $-P_1$ to P_2. (Fig. 1). It consists of three rods of the same cross sections' area S and of the same material with elastic module E. The rods can experience only uniaxial tensile/compressive deformation.

Due to symmetry the strains and stresses in rods 2 and 3 are identical: $\varepsilon_2 = \varepsilon_3$, $\sigma_2 = \sigma_3$. The strains of rods 1 and 2 are connected by the relation: $\varepsilon_1 = 2\varepsilon_2$. The stresses in the rods satisfy the equilibrium equation: $\sigma_1 + \sigma_2\sqrt{2} = p$, where $-p_1 \leqslant p \leqslant p_2$, $p = P/S$, $p_1 = P_1/S$, $p_2 = P_2/S$. In elastic state $\sigma_1^{\mathrm{e}} = 2\sigma_2^{\mathrm{e}} = p(2-\sqrt{2})$. Let us determine the shakedown conditions for various mechanical properties of the rod's material supposing that rods 2, 3 remain elastic.

(1) *Perfect plasticity*. Assume firstly that the rods are of an elastic perfectly plastic material with the limit stress k in tension/compression. The yield condition of the rod 1 is $F = |\boldsymbol{\sigma}_1| - k \leqslant 0$. In order that rod 1 would be plastic the bounds of the load p should satisfy the inequalities:

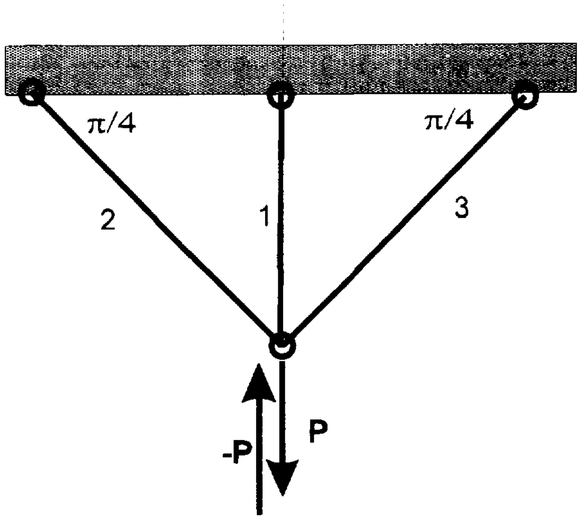

Fig. 1.

$$1+\sqrt{2}/2 \leqslant p_1/k \quad \text{or} \quad p_1/k < 1+\sqrt{2}/2 \leqslant p_2/k. \tag{18}$$

Rods 2, 3 are elastic if $p_2/k < 2+\sqrt{2}$. According to the Melan theorem shakedown takes place if such a constant residual stress σ^r exists that the inequality $|p(2-\sqrt{2})+\sigma^r|-k < 0$ holds. This inequality will be satisfied if

$$-k+p_1(2-\sqrt{2}) < \sigma^r < k-p_2(2-\sqrt{2}). \tag{19}$$

A residual stress σ^r satisfying (19), exists if the following inequality is valid

$$p_1+p_2 < k(2+\sqrt{2}). \tag{20}$$

This is a sufficient condition of elastic shakedown under the conditions taken. If this condition is violated, shakedown is impossible. In this case rod 1 experiences non decreasing alternating plastic deformation (plastic shakedown). Thus condition (20) is not only sufficient but necessary as well. If inequality (20) holds then $p_1/k < (1+\sqrt{2}/2)$. This implies that compressive stress in the rod 1 does not reach its limit value (see eqn $(18)_2$).

(2) *Isotropic strain hardening*. Let now the rods be of a material with unbounded isotropic strain hardening. Its yield function is taken in the form: $\Phi = |\sigma|-k_0-c\chi^{\alpha}$ where α, c, k_0 are material parameters and χ is a hardening parameter defined by the equation: $\dot{\chi} = |\dot{\varepsilon}^p|$, ($\varepsilon^p$ is the plastic part of rod 1 strain).

Let the range of the force P be again prescribed: $-p_1 \leqslant p \leqslant p_2$, $p_1 < p_2$. The problem is to determine the conditions under which rod 1 can shake down. To that end consider the set of yield conditions determined by different values of the hardening parameters χ as the set of yield conditions of elastic perfectly plastic bodies and determine the minimal adaptive yield condition. Because the condition of elastic shakedown for the structure of a perfectly plastic material is $p_1+p_2 < k(2+\sqrt{2})$ the value of the limit stress corresponding to the minimal adaptive yield surface is $k = (p_1+p_2)(1-\sqrt{2}/2)$

Thus, the minimal, adaptive yield function is $F_0 = |\sigma|-(p_1+p_2)(1-\sqrt{2}/2)$. According to Section 9 shakedown takes place if $\Phi \leqslant F_0$. This inequality provides a lower estimation to the limit value of the hardening parameter, χ^*.

$$\chi^* \geqslant \left(\frac{(p_1+p_2)(1-\sqrt{2}/2)-k_0}{c}\right)^{1/\alpha}. \quad (21)$$

A direct examination confirms that the right side of eqn (21) provides the actual value of the hardening parameter corresponding to the LYC.

(3) *Kinematic strain hardening*. Assume now that the rods are of a material with kinematic strain hardening. The yield function is taken in the form: $\Phi = |\sigma - c\varepsilon^p|\varepsilon^p|^\alpha| - k_0$. In that case, the developed methods again allow to obtain a lower estimation of the plastic strain ε^p which should be accumulated in rod 1 in order to shakedown was possible.

According to statement 2 of Section 8 shakedown is impossible if the yield surface $\Phi = 0$ is embraced by the minimal adaptive yield surface $F_0 = 0$, that is, if $\Phi > F_0$. Hence shakedown is possible if $\Phi \leqslant F_0$. This is a necessary condition for shakedown. A comparison of Φ with F_0 provides the minimal value of the plastic strain rod 1 under which the inequality $\Phi \leqslant F_0$ holds. This value is given by the right side of eqn (21) again.

Acknowledgements—This research has been accomplished in framework of the grant from Israel Science Foundation.

REFERENCES

Comi, C. and Corigliano, A. (1991). Dynamic shakedown in elastoplastic structures with general variable constitutive laws. *Int. J. Plast.* **7**, 679–692.

Corigliano, A., Maier, G. and Pycko, S. (1995). Kinematic criteria of dynamic shakedown extended to nonassociative constitutive laws with saturation nonlinear hardening. *Rend. Mat. Ass. Lincei* **9**(6), 55–64.

Druyanov, B. and Roman, I. (1995). Shakedown theorems extended to elastic non perfectly plastic bodies. *Mech. Res. Comm.* **22**, 571–576.

Gokhefeld, D. and Sadakov, O. (1995). Steady cyclic state of structure: methods of its direct determination. In: *Inelastic Response of Structures Under Variable Loads* (eds Z. Mroz, D. Weichert, and S. Dorosz), pp. 449–463.

Hachemi, A. and Weichert, D. (1992). An extension of the shakedown theorem to a certain class of inelastic materials with damage. *Arch. Mech.* **44**, 491–498.

Halphen, B. and Nguyen, Q. S. (1975). Sur le matériaux standard généralisés. *J. Mécanique* **14**, 39–63.

Ju, J. W. (1989). On energy-based coupled elastoplastic damage theories: constitutive modeling and computational aspects. *Int. J. Solids Struct.* **25**(7) 803–833.

Koiter, W. T. (1960). General theorems of elastic–plastic solids. In: *Progress in Solids Mechanics* (eds I. N. Sneddon and R. Hill), pp. 167–221, North Holland, Amsterdam.

Konig, J. A. (1982). On some recent developments in the shakedown-theory. *Adv. Mech.* **5**, 237–258.

Konig, J. A. and Siemaszko, A. (1988). Strain hardening effects in shakedown processes. *Ing. Arch.* **58**, 58–66.

Maier, G. (1969). Shakedown theory in elastoplasticity with associated and nonassociated flow laws: a finite element, linear programming approach. *Meccanica* **4**, 1–11.

Maier, G. (1973). A shakedown matrix theory allowing for work hardening and second-order geometric effects. In: *Foundation of Plasticity* (Edited by A. Sawczuk), Vol. 1, pp. 417–433. Noordhoff, Leyden.

Maier, G. and Novati, G. (1990a). Dynamic shakedown and bounding theory for a class of nonlinear hardening discrete structural models. *Int. J. Plast.* **6**, 551–572.

Maier, G. and Novati, G. (1990b). A shakedown and bounding theory allowing for nonlinear hardening and second-order geometric effects. In: *Inelastic Solids and Structures* (eds M. Kleiber and J. A. Konig), pp. 451–472. Pineridge Press, Swansea.

Mandel, J. (1976). Adaptation d'une structure plastique ecrouissable et approximations. *Mech. Res. Comm.* **3**, 483–488.

Martin, J. B. (1975). *Plasticity: Fundamentals and General Results*. MIT Press, Cambridge, MA.

Melan, E. (1938). Zur Plastizitat des raumlichen Kontinuums. *Ing. Arch.* **9**, 115–126.

Nayroles, B. and Weichert, D. (1993). La notion de sanctuaire d'élasticité et d'adaptation des structures. C. R. de l'Acad. Sci. Paris, **316**, (II) (Méc. des solides), 1493–1498.

Polizzotto, C., Borino, G., Caddemi, S. and Fuschi, P. (1991). Shakedown problems for material models with internal variables. *Eur. J. Mech.* A/Solids **10**, 621–639.

Polizzotto, C. (1994). Steady states and sensitivity analysis in elastic plastic structures subjected to cyclic loading. *Int. J. Solids Struct.* **31**, 953–970.

Ponter, A. R. S. (1975). A general shakedown theorem for elastic/plastic bodies with work hardening. *Proc. SMIRT-3*, London, paper L5/2.

Pycko, S. and Maier, G. (1995). Shakedown theorems for some classes of nonassociated hardening elastic-plastic material models. *Int. J. Plast.* **11**, 367–395.

Stein, E., Zhang, G. and König, J. A. (1992). Shakedown with nonlinear strain-hardening including structural computation using finite element method. *Int. J. Plast.* **8**, 1–31.

Weichert, D. and Gross-Weege, J. (1988). The numerical assessment of elastic–plastic sheets under variable mechanical and thermal loads using a simplified two-surface yield condition. *Int. J. Mech. Sci.* **30**, 757–767.